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One-point functions in finite volume/temperature: a case study

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Abstract

We consider finite volume (or equivalently, finite temperature) expectation values of local operators in integrable quantum field theories using a combination of numerical and analytical approaches. It is shown that the truncated conformal space approach, when supplemented with a recently proposed renormalization group, can be sufficiently extended to the low-energy regime that it can be matched with high precision by the low-temperature expansion proposed by Leclair and Mussardo. Besides verifying the consistency of the two descriptions, their combination leads to an evaluation of expectation values which is valid to a very high precision for all volume/temperature scales. As a side result of the investigation, we also discuss some unexpected singularities in the framework recently proposed by Pozsgay and Takács for the description of matrix elements of local operators in finite volume, and show that while some of these singularities are resolved by the inclusion of the class of exponential finite size corrections known as μ-terms, these latter corrections themselves lead to the appearance of new singularities. We point out that a fully consistent description of finite volume matrix elements is expected to be free of singularities, and therefore a more complete and systematic understanding of exponential finite size corrections is necessary.

1 Introduction

Finite temperature expectation values of local observables are important for numerous applications of quantum field theory. In $1+1$ dimensional integrable quantum field theories, they have been the subject of intensive studies recently. In [1], Leclair and Mussardo conjectured a low-temperature expansion which uses two ingredients: the thermodynamic Bethe Ansatz [2] as applied to integrable quantum field theories [3], and the form factor bootstrap program [4, 5, 6]. Although a partial proof of the series was given by Saleur in [7], it was not until the recent introduction of the finite volume form factor framework by Pozsgay and Takács [8, 9] that a systematic construction of the series itself was made possible. The finite volume form factor formalism even led to an explicit derivation and further generalization of the series to all orders [10]. The series itself has found applications in the investigations of one-dimensional quantum gases [11, 12, 13]. The finite volume form factor formalism, on the other hand, was extended to two-point functions at finite temperature [14, 15] and a method to determine the expansion to any given order was developed in [16, 17].

The aim of the present work is to address some open issues, regarding both the finite volume form factor formalism and the determination of thermal one-point functions. Regarding the latter, we make use of the equivalence between a system at finite temperature T and in a finite volume $L = 1/T$, we consider the one-point function in finite volume, as determined using the following methods:

- the Leclair-Mussardo series;
- the thermodynamic Bethe Ansatz (TBA), which is equivalent to the Leclair-Mussardo (LM) series applied to the trace of the energy-momentum tensor;
- and the truncated conformal space approach (TCSA), introduced in [18].

The LM series is a low-energy expansion, valid for large values of L, while the TCSA is a non-perturbative extension of the ultraviolet perturbation theory, which is valid for small values of L. Therefore care must be taken to find a regime where these approaches can be matched. This is facilitated by applying a recently introduced TCSA renormalization group method [19, 20, 21] to the evaluation of one-point functions, in order to improve the precision of their TCSA evaluation for larger values of the volume.

In addition to the thermal/finite volume expectations values, this work also discusses a separate, but related issue that we found when testing for the consistency of the TCSA and the bootstrap form factors. Namely, due to singular behaviour in the density of states, the finite volume form factors predicted by the formulae in [8, 9] show some singularities. It turns out that these can be resolved by including some of the exponential finite size corrections, the so-called μ-terms along the lines of [22].

For concrete calculations we chose the T_2 model which is an integrable perturbation of the minimal conformal field theory $\mathcal{M}_{2,7}$ by its relevant perturbation $\Phi_{1,3}$. This was motivated by the fact that this is one of the simplest theories in which there is an additional non-trivial primary field besides the perturbing one, so this provides a non-trivial application of the LM series in the sense that the series for this operator is not a consequence of the TBA.

The outline of the paper is as follows. In section 2 we briefly recall relevant facts about the LM series and TBA. In section 3 we give a derivation of the cut-off dependence of expectation values in TCSA, since we need the explicit exponents for our calculations. While the

extrapolation has already been used on at least two occasions [20, 23], the theory behind the exponents has never been exposed and our derivation fills this gap. Section 4 is devoted to the comparison of the one-point function obtained from the TCSA on the one hand, and the LM series (or, equivalently, from the TBA in the case of the trace of the energy-momentum tensor) on the other. In section 5 we digress a little to discuss the issue of singularities in the finite volume form factor formalism and their resolution by the inclusion of μ-term corrections. Section 6 is reserved for the conclusions. There are two appendices: Appendix A gives the relevant details about TCSA and its application to the T_2 model, while Appendix B is a brief summary of its scattering theory and form factors.

2 One-point functions at finite temperature

2.1 The Leclair-Mussardo series

2.1.1 The LM series itself

The conjecture for the finite volume expectation value of a local operator by LeClair and Mussardo [1] takes the following form in the T_2 model:

$$\begin{aligned}\langle\mathcal{O}\rangle_L &= \sum_{n,m=0}^{\infty}\frac{1}{n!m!}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1}{2\pi}\cdots\frac{\mathrm{d}\theta_n}{2\pi}\frac{\mathrm{d}\tilde{\theta}_1}{2\pi}\cdots\frac{\mathrm{d}\tilde{\theta}_n}{2\pi}\prod_{i=1}^{n}\frac{1}{1+e^{\varepsilon_1(\theta_i)}}\prod_{j=1}^{m}\frac{1}{1+e^{\varepsilon_2(\tilde{\theta}_j)}}\\ &\quad\times F^{\mathcal{O}}_{2n,2m,c}\left(\theta_1,\ldots,\theta_n,\tilde{\theta}_1,\ldots,\tilde{\theta}_m\right)\end{aligned} \tag{2.1}$$

where θ_i, $\tilde{\theta}_i$ are rapidities of the particles with mass m_1 and m_2. The ε_a are the pseudo-energy functions that satisfy the TBA integral equation

$$\varepsilon_a(\theta) = m_a L\cosh\theta + \sum_b\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi}\varphi_{ab}\left(\theta-\theta'\right)\log\left(1+e^{-\varepsilon_b(\theta')}\right) \tag{2.2}$$

with φ_{ab} the derivative of the phase shift of the S-matrix, and $F^{\mathcal{O}}_{2n,2m,c}$ is the connected form factor defined by

$$\begin{aligned}&F^{\mathcal{O}}_{2n,2m,c}\left(\theta_1,\ldots,\theta_n,\tilde{\theta}_1,\ldots,\tilde{\theta}_m\right) =\\ &\text{finite part of } F^{\mathcal{O}}_{\underbrace{2\ldots2}_{m}\underbrace{1\ldots1}_{n}\underbrace{1\ldots1}_{n}\underbrace{2\ldots2}_{m}}\Big(\tilde{\theta}_m+i\pi+\epsilon_{m+n},\ldots,\tilde{\theta}_1+i\pi+\epsilon_{m+1},\\ &\theta_n+i\pi+\epsilon_{n+1},\ldots,\theta_1+i\pi+\epsilon_1,\theta_1,\ldots,\theta_n,\tilde{\theta}_1,\ldots,\tilde{\theta}_m\Big)\end{aligned} \tag{2.3}$$

where the finite part is taken by expanding the ϵ-dependence of the form factor expression for small values of the ϵ-s and omitting all terms which blow up when any of the ϵ_k is taken to 0 (including terms that depend on the ratios of the ϵ_k) [1].

2.1.2 Evidence for the LM series: TBA and finite volume form factors

Leclair and Mussardo showed that for the trace of energy momentum tensor their series coincides with the result derived from the TBA by Zamolodchikov [3], which we briefly summarize below specialized to the T_2 model.

Since T_2 is the perturbation of $\mathcal{M}_{2,7}$ minimal model by $\Phi_{1,3}$ operator, the trace of the stress energy tensor is given by[1]

$$\Theta = T^{\mu}_{\mu} = 2\pi\lambda(2h_{1,3}-2)\Phi_{1,3} \tag{2.4}$$

where λ is the coupling constant and $h_{1,3}$ the conformal weight of the operator. The ground state energy in finite volume is given by

$$E_0(L) = -\sum_a \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m_a \cosh\theta \log\left(1+e^{-\varepsilon_a(\theta)}\right)$$

where the ε_a solve (2.2). The vacuum expectation value of Θ can be expressed by the following derivative of $E_0\left(L\right)$

$$\begin{aligned}\langle\Theta\rangle_L &= 2\pi\frac{1}{L}\frac{d}{dL}\left[LE_0\left(L\right)\right] = 2\pi\left[\frac{E_0\left(L\right)}{L}+\frac{dE_0\left(L\right)}{dL}\right] \\ &= \sum_a m_a \int_{-\infty}^{\infty} d\theta \frac{1}{1+e^{\varepsilon_a(\theta)}}\left[\cosh\theta\partial_L\varepsilon_a\left(\theta\right)-\frac{1}{L}\sinh\theta\partial_\theta\varepsilon_a\left(\theta\right)\right]\end{aligned}$$

where the derivatives of the TBA pseudo-energy solve the following linear equations

$$\partial_L\varepsilon_a\left(\theta\right) = m_a\cosh\theta + \sum_b \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\varphi_{ab}\left(\theta-\theta'\right)\frac{1}{1+e^{\varepsilon_b(\theta')}}\partial_L\varepsilon_b\left(\theta'\right)$$

$$\partial_\theta\varepsilon_a\left(\theta\right) = m_aL\sinh\theta + \sum_b \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\varphi_{ab}\left(\theta-\theta'\right)\frac{1}{1+e^{\varepsilon_b(\theta')}}\partial_\theta\varepsilon_b\left(\theta'\right)$$

By expanding the solution to these equations for large L, one obtains the series (2.1) [1]. Extending the argument, Saleur proved the equivalence of the TBA and the LM series for one-point functions of conserved charge densities [7].

For general operators where a TBA construction is not known, the LM series was demonstrated to be true up to (and including) three particle terms in [9], and a proof to all orders was presented in [10]. We remark that these results were proven using the finite volume form factor formalism introduced in [8, 9], and the proofs relies on the expression of finite volume diagonal matrix elements (up to exponential corrections in the volume), which is still only a conjecture, albeit well-supported both by analytical arguments [7, 9] and numerical data [9].

3 One-point functions from TCSA using RG extrapolation

3.1 Cut-off dependence of the vacuum expectation value

For a numerical evaluation of the expectation values, we used the truncated conformal space approach (TCSA) developed by Yurov and Zamolodchikov in [18]; for the technical details we refer the reader to appendix A.

[1]The stress energy tensor is normalized following the conventions of Zamolodchikov's paper [3].

In order to test the exponentially decaying finite volume corrections predicted by the LM series, we need to improve the accuracy of the TCSA. This can be achieved by extrapolating in the cut-off (truncation level) dependence using TCSA RG, which was first applied in [20], and later also in [23]. However, the way to derive the necessary exponents has not yet been presented; therefore here we give the necessary details below so that we can apply it to our case, and also in order to facilitate future applications of the method.

The derivation of the cut-off dependence below uses the general ideas presented in [21] where a similar method was used to obtain the running coupling constants. We parametrize the cut-off $e_{\rm cut}$ by the left descendant quantum number of the highest vectors included in the truncated space. In the T_2 model, this is the same for all modules in the Hilbert space as all the differences between the primary dimensions are smaller than 2. The vacuum expectation value of the operator $\mathcal{O}$, when cutting TCSA at level n is:

$$E(n) = \frac{Q(n)}{\mathcal{N}(n)}$$

$$Q(n) = \langle \min |\, P_n \mathcal{O}(0,0) P_n e^{-\lambda \int d^2 z P_n V(z,\bar{z}) P_n} | \min \rangle$$

where $z = \tau - ix$, V is the perturbing operator, which has conformal spin zero (i.e. $h_V = \bar{h}_V$) to ensure translational invariance; to simplify the exposition, we also assume that $h_{\mathcal{O}} = \bar{h}_{\mathcal{O}}$. All operator products are assumed to be time ordered, P_n is the projector on states with descendant numbers less than or equal to n and $|\min\rangle$ is the lowest weight primary state, whose perturbation becomes the vacuum of the massive theory (for T_2 this is the state created by $\Phi_{1,3}$). The normalization factor is

$$\mathcal{N}(n) = \langle \min |\, e^{-\lambda \int d^2 z P_n V(z,\bar{z}) P_n} | \min \rangle$$

Our aim is to calculate the dependence on n, i.e. the difference $Q(n) - Q(n-1)$. To simplify the subsequent formulae, let us introduce the following notation for the past and future evolution operators corresponding to the cut-off Hamiltonian:

$$U_{\pm}^{(n)}(\lambda) = T \exp\left(-\lambda \int_{\tau' \gtrless 0} d^2 z' P_n V(z', \bar{z}') P_n \right)$$

so we can make time ordering explicit

$$Q(n) \;\; = \;\; \langle \min |\, U_{+}^{(n)}(\lambda) P_n \mathcal{O}(0,0) P_n U_{-}^{(n)}(\lambda) | \min \rangle$$

We assume that n is large, and so the states at descendant level n have energy much higher than the mass gap m:

$$\frac{4\pi n}{L} \gg m \tag{3.1}$$

When this condition is satisfied, we can evaluate their contribution to first order in λ. Defining the projector on the level n descendants as $\tilde{P}_n = P_n - P_{n-1}$ and denoting

$$\Delta_n V(z,\bar{z}) = P_{n-1} V(z,\bar{z}) \tilde{P}_n + \tilde{P}_n V(z,\bar{z}) P_{n-1} + \tilde{P}_n V(z,\bar{z}) \tilde{P}_n$$

the $\tau < 0$ time-evolution operator can be expanded as

$$\begin{aligned} U_-^{(n)}(\lambda) &= T\exp\left(-\lambda\int_{\tau<0} d^2z\,(P_{n-1}V(z,\bar z)P_{n-1}+\Delta_n V(z,\bar z))\right) = \\ & T\left\{\left[1-\lambda\int_{\tau<0} d^2z\Delta_n V(z,\bar z)\right]\exp\left(-\lambda\int_{\tau'<0} d^2z' P_{n-1}V(z',\bar z')P_{n-1}\right)\right\}+O(\lambda^2) \\ &= \left[1-\lambda\int_{\tau<0} d^2z\Delta_n V(z,\bar z)\right]U_-^{(n-1)}(\lambda)+O(\lambda^2) \end{aligned}$$

and similarly for $U_+^{(n)}(\lambda)$. This yields

$$\begin{aligned} Q(n)-Q(n-1) &= \langle\min|\left\{U_+^{(n-1)}(\lambda)\left[1-\lambda\int_{\tau>0} d^2z V(z,\bar z)\tilde P_n\right]\left(P_{n-1}+\tilde P_n\right)\right. \\ &\qquad \left.\times\mathcal{O}(0,0)\left(P_{n-1}+\tilde P_n\right)\left[1-\lambda\int_{\tau<0} d^2z\tilde P_n V(z,\bar z)\right]U_-^{(n-1)}(\lambda)\right\}|\min\rangle \\ &\quad -\langle\min|\,U_+^{(n-1)}(\lambda)P_{n-1}\mathcal{O}(0,0)P_{n-1}U_-^{(n-1)}(\lambda)|\min\rangle+O(\lambda^2) \end{aligned}$$

where we used that as a consequence of

$$P_{n-1}\tilde P_n = \tilde P_n P_{n-1} = 0$$

$\tilde P_n$ commutes with $U_\pm^{(n-1)}(\lambda)$, and also the relations

$$\begin{aligned} \tilde P_n\,|\min\rangle &= 0 \\ P_{n-1}\,|\min\rangle &= |\min\rangle \end{aligned}$$

Exploiting invariance under time reflection $\tau\to-\tau$

$$Q(n)-Q(n-1) = -2\lambda\int_{\tau<0} d^2z\,\langle\min|\,U_+^{(n-1)}(\lambda)\,\mathcal{O}(0,0)\tilde P_n V(z,\bar z)\,U_-^{(n-1)}(\lambda)\,|\min\rangle+O(\lambda^2)$$

Applying the exponential map (A.1) and writing $w=re^{i\varphi}$

$$\int_{\tau<0} d^2z = \left(\frac{L}{2\pi}\right)^2\int_0^1\frac{dr}{r}\int_0^{2\pi} d\varphi$$

yields

$$\begin{aligned} &Q(n)-Q(n-1) = \qquad (3.2)\\ &-2\lambda\left(\frac{2\pi}{L}\right)^{2h_{\mathcal{O}}+2h_V-2}\int_0^1\frac{dr}{r}\int_0^{2\pi} d\varphi\, r^{2h_V}\,\langle\min|\,U_+^{(\infty)}(\lambda)\,\mathcal{O}(1,1)\tilde P_n V(w,\bar w)\,U_-^{(\infty)}(\lambda)\,|\min\rangle \end{aligned}$$

where we used that to leading order one can take $n\to\infty$ in the time evolution exponentials. Using the operator product expansion (OPE) in conformal field theory

$$\mathcal{O}(1,1)V(w,\bar w) = \sum_A\frac{C_{\mathcal{O}V}^A\,A(1,1)}{(1-w)^{h_{\mathcal{O}}+h_V-h_A}(1-\bar w)^{\bar h_{\mathcal{O}}+\bar h_V-\bar h_A}} \qquad (3.3)$$

where A runs over the set of scaling fields and the C are the operator product structure constants.

As described in Appendix A, due to the fact that the perturbing operator satisfied $h_V = \bar{h}_V$, the TCSA Hilbert space decomposes into sectors with a given value of the conformal spin $L_0 - \bar{L}_0$, which is in fact a consequence of translational invariance. Since the vacuum state is translationally invariant it has zero conformal spin. Therefore all intermediate states contributing to the matrix element have spin zero, and so the operators A contributing to the cutoff dependence also have zero conformal spin, i.e. $h_A = \bar{h}_A$ as well. Substituting (3.3) into (3.2) and using the expansions

$$\begin{aligned}\frac{1}{(1-w)^\alpha} &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} w^n \\ \frac{1}{(1-\bar{w})^\alpha} &= \sum_{\bar{n}=0}^{\infty} \frac{\Gamma(\alpha+\bar{n})}{\Gamma(\alpha)\Gamma(\bar{n}+1)} \bar{w}^{\bar{n}}\end{aligned}$$

the $(n,\bar{n})$ term just gives the contribution to the $\mathcal{O}V$ two-point function from intermediate states at level $(n,\bar{n})$. Due to the angular integral the only contributing terms are those in which $n = \bar{n}$, which is another manifestation of translational invariance. The exponentials ensure that the contribution of a given operator A in the OPE will be proportional to the exact finite volume vacuum expectation value in the perturbed theory

$$\langle \mathrm{min}|\, U_+^{(\infty)}(\lambda) A(w,\bar{w}) U_-^{(\infty)}(\lambda)\, |\mathrm{min}\rangle \Big|_{w=\bar{w}=1}$$

This captures the non-perturbative infrared physics, while perturbation theory applies in the UV regime and describes the dependence on n (provided that n is large enough).

Therefore a given operator A contributes to $Q(n) - Q(n-1)$ a term proportional to

$$\int_0^1 \left(\frac{\Gamma(\alpha_A+n)}{\Gamma(\alpha_A)\Gamma(n+1)}\right)^2 r^{2n-1+2h_V} dr = \frac{1}{2(n+h_V)} \left(\frac{\Gamma(\alpha_A+n)}{\Gamma(\alpha_A)\Gamma(n+1)}\right)^2$$

where

$$\alpha_A = h_{\mathcal{O}} + h_V - h_A$$

and all factors independent of n were omitted. To evaluate the leading behaviour for large n, one can use Stirling's formula

$$\Gamma(x+1) \approx \sqrt{2\pi x} x^x e^{-x} \left(1 + O(1/x)\right)$$

and the relation

$$(n+a)^{(n+a)} \approx n^{n+a} e^a \left(1 + O(1/n)\right)$$

which results in

$$Q(n) - Q(n-1) \propto \sum_A K_A n^{2\alpha_A - 3} \left(1 + O(1/n)\right)$$

where K_A are n-independent coefficients. For large n, this gives the differential equation[2]

$$\frac{dQ(n)}{dn} \propto \sum_A K_A n^{2\alpha_A - 3} \left(1 + O(1/n)\right)$$

[2]We remark that the difference equation is also easy to solve and the difference from the differential equation is numerically important for small n. On the other hand, for the relatively high values of the cutoff the difference between the two is insignificant.

Solving the equation with the initial condition at $n = \infty$, the result is

$$Q(n) = Q(\infty) + \sum_A \tilde{K}_A n^{2\alpha_A - 2} \left(1 + O(1/n)\right) + O(\lambda^2)$$

The leading corrections in the descendant level are of order $1/n$ (as indicated), arising from omitting h_V besides n, and also from the approximations used in Stirling's formula.

For the complete expectation value $E(n)$ one must also compute the running of the normalization factor $\mathcal{N}(n)$. This is the same as $Q(n)$ for the choice when $\mathcal{O}$ is the identity, and so $A = V$ is the only operator in the OPE. Therefore the relevant exponent α is $h_{\mathcal{O}} + h_V - h_A = 0$ and there is no $w, \bar{w}$ dependence in the OPE, and so $\mathcal{N}(n)$ does not run to first order in λ:

$$\frac{d\mathcal{N}(n)}{dn} = O(\lambda^2)$$

Supposing that the expectation value we compute is convergent in TCSA (i.e. $2\alpha_A - 2 < 0$ for all A), the end result for the matrix element is:

$$E(n) = E(\infty) + \sum_A K_A n^{2\alpha_A - 2} \left(1 + O(1/n)\right) + O(\lambda^2) \tag{3.4}$$

The issue now is whether this calculation is self-consistent, i.e. whether the $O(\lambda^2)$ terms are smaller than the $O(\lambda)$ terms we evaluated. Since they are rather complicated to evaluate, we rather use a "proxy" condition which is to require that the first order correction itself is small when compared to $E(\infty)$. One can neglect the exponentially small difference between the infinite and finite volume expectation values of $\mathcal{O}$:

$$E(\infty) - \langle \mathcal{O} \rangle_\lambda \sim O\left(e^{-mL}\right)$$

and assuming that A is a primary field (so it transforms homogeneously under the exponential mapping) we get

$$\langle \mathrm{min}|\, U_+^{(\infty)}(\lambda) A(w, \bar{w}) U_-^{(\infty)}(\lambda)\, |\mathrm{min}\rangle \Big|_{w=\bar{w}=1} = \left(\frac{L}{2\pi}\right)^{2h_A} \langle A \rangle_\lambda + O\left(e^{-mL}\right)$$

in terms of the infinity volume vacuum expectation value $\langle A \rangle_\lambda$. Up to some dimensionless numerical coefficients, this leads to

$$\lambda \left(\frac{2\pi}{L}\right)^{2h_{\mathcal{O}}} \left(\frac{2\pi}{L}\right)^{2h_V} 2 \left(\frac{L}{2\pi}\right)^2 \left(\frac{L}{2\pi}\right)^{2h_A} \langle A \rangle_\lambda \, n^{2\alpha_A - 2} \ll \langle \mathcal{O} \rangle_\lambda$$

Simple dimensional analysis gives

$$\lambda \propto m^{2-2\Delta_V} \quad , \quad \langle A \rangle_\lambda \propto m^{2-2\Delta_A} \quad , \quad \langle \mathcal{O} \rangle_\lambda \propto m^{2-2\Delta_{\mathcal{O}}}$$

up to numerical coefficients. This leads to

$$\left(\frac{2\pi n}{mL}\right)^{2\alpha_A - 2} \ll 1 \tag{3.5}$$

i.e.

$$\frac{mL}{2\pi n} \ll 1$$

which is the same condition as (3.1). This is indeed consistent with what we have observed in all of our practical calculations performed so far: the first order extrapolation breaks down in large enough volume (the precise value of which depends on the model considered); and the higher the value of the cut-off, the larger is the critical value of the volume where the break-down occurs.

3.2 Remarks

First of all note that to leading order all the calculation is unchanged if we consider a general matrix element instead of a vacuum expectation value, as long as the cut-off is chosen so that the states between which the matrix element is taken are included in the truncated Hilbert space. The only difference is that instead of the vacuum matrix element of operator A one obtains the appropriate excited state finite-volume matrix element as coefficient. Therefore, while the exponent of the cutoff dependence is universal, its coefficient depends on the matrix element considered.

If the operator is such that its matrix elements are not convergent with increasing cutoff, the operator must be renormalized by subtracting the divergent parts. These are always proportional to operators with lower scaling dimensions, as long as the perturbation is relevant (ie. the theory is renormalizable). This is true because the condition for divergence is

$$2\alpha_A - 2 > 0$$

which leads to

$$2(h_{\mathcal{O}} + h_V - h_A) - 2 > 0$$

and since in a renormalizable theory the perturbing operator is relevant ($h_V \leq 1$), a divergent term can only be obtained if for some A

$$h_A < h_{\mathcal{O}}$$

In this case, a renormalized operator $[\mathcal{O}]$ must be defined by subtracting from $\mathcal{O}$ a counter term proportional to A. This is consistent with the general theorems of renormalization derived in the framework of the standard Feynman perturbation expansion in renormalizable quantum field theory.

Finally, note that the cut-off dependence can also be obtained by a simple scaling argument, by examining the operator product expansion:

$$\int d^2 z V\mathcal{O} \sim \sum_A C_A A$$

The energy dependence of the coefficient is given by simple dimensional analysis as

$$C_A \propto [\text{energy}]^{-2+2\alpha_A} \quad , \quad \alpha_A = h_{\mathcal{O}} + h_V - h_A$$

The typical energy of (zero-spin) states at descendant level n is

$$\frac{4\pi n}{L}$$

If n is high, then this energy is much larger than the mass scale m, so by usual rules of ultraviolet perturbation theory the dependence on the mass scale can be neglected to leading order, which means that the above energy is the only scale in the calculation. Therefore the contribution from level n states to leading order takes the form

$$C_A \propto \left(\frac{4\pi n}{L}\right)^{-2+2\alpha_A}$$

which also gives the correct volume dependence as well.

In TCSA everything is measured in units of the mass scale m, so the relevant dimensionless number which characterizes the magnitude of the first order correction is

$$c_A = C_A m^{2-2\alpha_A} \propto \left(\frac{4\pi n}{mL}\right)^{-2+2\alpha_A}$$

in agreement with (3.5).

3.3 Some particular models

3.3.1 Ising model in a magnetic field

This is the model with the famous E_8 scattering theory obtained by Zamolodchikov [24]. The Hamiltonian is given by

$$H = H_* + h \int dx\, \sigma$$

where σ is the spin-field with $h = \bar{h} = 1/16$. For the matrix elements of σ itself, the most relevant operators occurring in $\sigma\sigma$ are

$$\begin{aligned} A = 1 \quad &: \quad h_A = 0 \\ A = \epsilon \quad &: \quad h_A = 1/2 \end{aligned}$$

and so the corresponding leading cut-off dependencies are

$$n^{-7/4} \qquad , \qquad n^{-11/4}$$

where the second exponential is of the same order as the (neglected) $1/n$ corrections to the first. In fact, an extrapolation using the first exponent gives an excellent result for the vacuum–two-particle matrix element, as demonstrated in [20].

3.3.2 Sine-Gordon model

This is a perturbation of a massless free boson ($c = 1$ conformal field theory) with the Hamiltonian

$$H = \int dx\, \frac{1}{2}\left((\partial_t \Phi)^2 + (\partial_x \Phi)^2\right) - \lambda \int dx\ :\cos\beta\Phi:$$

and so in this case $V =: \cos\beta\Phi :$.

At first sight our derivation of the cut-off dependence is not really applicable to the sine-Gordon model. As the spectrum of primary fields is not bounded from above, with increasing cut-off $e_{\rm cut}$ new Fock modules enter the Hilbert space [25]. Therefore there is no simple relation between the cut-off and the level of the highest descendants in the Hilbert space. Nevertheless, it turns out that in the case discussed below only two operators contribute in the operator product, and therefore there is a one-to-one relation between $e_{\rm cut}$ and the highest descendant level n included in the two contributing Fock modules, which is enough for our derivation to apply.

Let us investigate the operator

$$\mathcal{O} =: e^{i\beta\Phi} :$$

We have

$$h_{\mathcal{O}} = h_V = h = \frac{\beta^2}{8\pi}$$

Using the operator product of exponential fields

$$: e^{i\alpha\Phi(z,\bar{z})} :: e^{i\beta\Phi(w,\bar{w})} := |z-w|^{-\frac{\alpha\beta}{2\pi}} : e^{i\alpha\Phi(z,\bar{z})+i\beta\Phi(w,\bar{w})} :$$

the leading operators in the $\mathcal{O}V$ OPE are:

$$\begin{aligned} A = 1 &: h_A = 0 \\ A = \partial\Phi\bar{\partial}\Phi &: h_A = 1 \\ A = e^{2i\beta\Phi} &: h_A = 4h \end{aligned}$$

which lead to the following cut-off dependence:

$$n^{-2+4h} \quad , \quad n^{-4+4h} \quad , \quad n^{-2-4h}$$

Note that for $8h > 1$, the $1/n$ corrections to the first term have a larger exponent than the third.

The extrapolation with only the leading term, when combined with a numerical renormalization group, gives excellent results in the attractive regime, as demonstrated in [23].

3.3.3 Scaling Lee-Yang model

This is the model for which TCSA was originally developed [18]. In this case, there is a single non-trivial primary field Φ. Choosing Φ as both the perturbing and measured operator:

$$\mathcal{O} = V = \Phi$$

one has

$$h_{\mathcal{O}} = h_V = h = -\frac{1}{5}$$

The operators occurring in the $\mathcal{O}V$ are

$$\begin{aligned} A = \Phi &: h_A = -\frac{1}{5} \Rightarrow \alpha_A = h \\ A = 1 &: h_A = 0 \Rightarrow \alpha_A = 2h \end{aligned}$$

and so the cut-off dependence is given by the terms

$$n^{-12/5} \quad , \quad n^{-14/5}$$

When applying this cut-off dependence to extrapolate TCSA measurements of the finite volume vacuum expectation value to infinite cut-off, it turned out that TCSA data are so detailed and accurate that we could also fit the subleading exponents

$$n^{-17/5} \quad , \quad n^{-19/5}$$

resulting from the $1/n$ corrections. These fits to the cutoff dependence are very accurate, and the resulting extrapolating matrix elements are shown in figure 3.1. These results also demonstrates that the RG improvement is really necessary to guarantee a proper matching between TCSA and the Leclair-Mussardo series (which is equivalent to the TBA in this particular case). Note that, as predicted by the argument in Section 3.2, the extrapolation becomes inaccurate in large volume. One expects that in this case it is necessary to evaluate the cutoff dependence to second order in perturbation theory, which woud require integrating a four point function in the conformal field theory, and is out of scope of the present work.

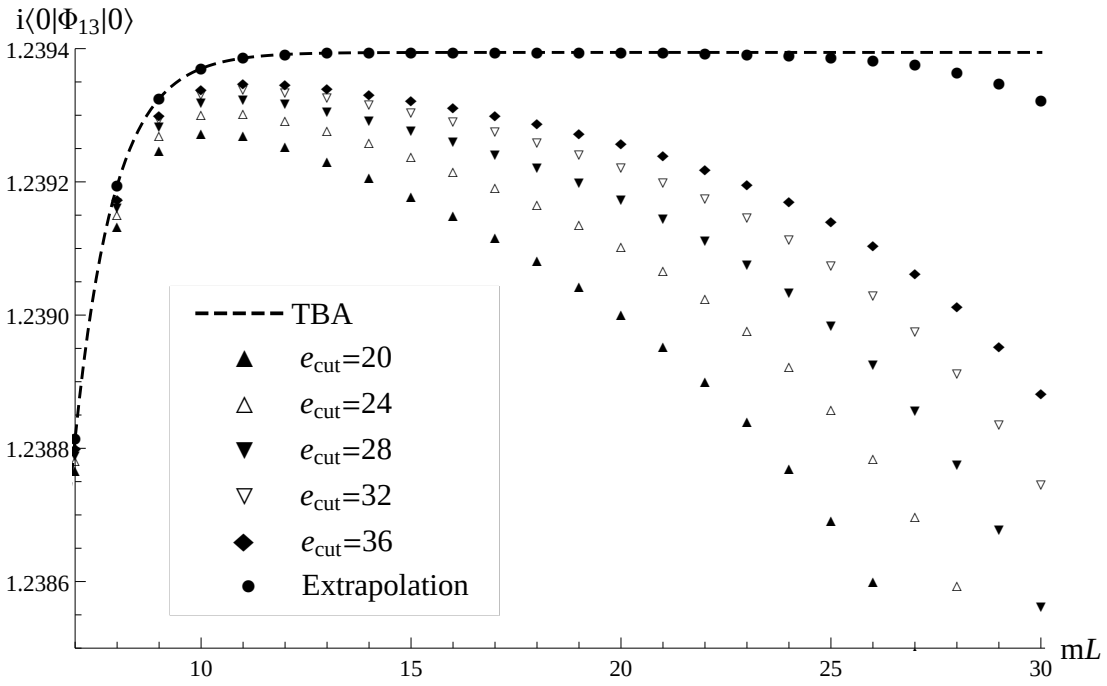

Figure 3.1: Extrapolation of the dimensionless vacuum expectation value $im^{2/5}\langle\Phi_{1,3}\rangle$ in the scaling Lee-Yang model, compared to the exact value computed from TBA.

3.3.4 T_2 model

The model itself is described in Appendices A and B. Let us first take the measured operator to be

$$\mathcal{O} = \Phi_{1,2}$$

Since $V = \Phi_{1,3}$, the operators occurring in the $\mathcal{O}V$ are

$$\begin{aligned} A = \Phi_{1,2} \quad &: \quad h_A = -\frac{2}{7} \Rightarrow \alpha_A = -\frac{3}{7} \\ A = \Phi_{1,4} \quad &: \quad h_A = -\frac{3}{7} \Rightarrow \alpha_A = -\frac{2}{7} \end{aligned}$$

and so the cut-off dependence is given by

$$n^{-18/7} \qquad , \qquad n^{-20/7}$$

For the case

$$\mathcal{O} = \Phi_{1,3}$$

the operators occurring in the $\mathcal{O}V$ are

$$\begin{aligned} A = 1 \quad &: \quad h_A = 0 \Rightarrow \alpha_A = -\frac{6}{7} \\ A = \Phi_{13} \quad &: \quad h_A = -\frac{3}{7} \Rightarrow \alpha_A = -\frac{3}{7} \end{aligned}$$

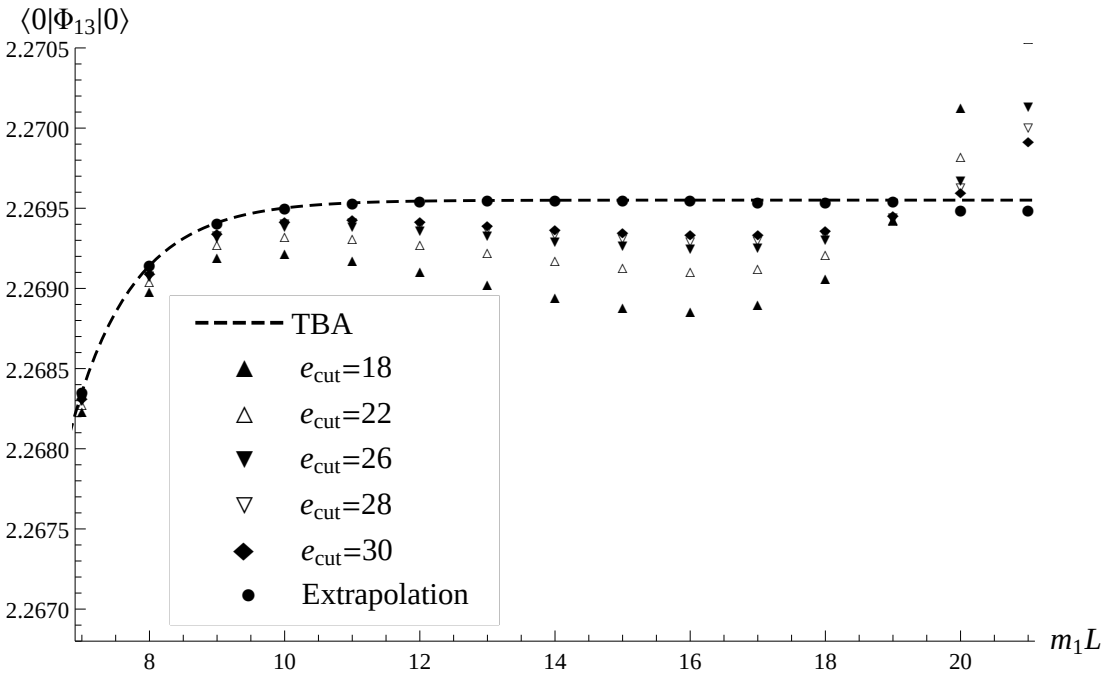

Figure 3.2: Extrapolation of the dimensionless vacuum expectation value $m^{6/7}\langle\Phi_{1,3}\rangle$ in the T_2 model, compared to the exact value computed from TBA.

and so the cut-off dependence is

$$n^{-20/7} \quad , \quad n^{-26/7}$$

The result of the extrapolation is for the perturbing operator $\Phi_{1,3}$ is shown in figure 3.2; again, the RG improvement is really necessary to guarantee a proper agreement.

4 Numerical comparison

Here we aim to compare the Leclair-Mussardo series to the RG-extrapolated TCSA results, for the operators $\Phi_{1,3}$ and $\Phi_{1,2}$. The expectation value of the $\Phi_{1,3}$ operator in the T_2 model is related to the trace of the stress-energy tensor (2.4), so its expectation value can be evaluated from TBA and it does not give us an independent validation for the LM series. However, the error in the numerical solution of the TBA equation (4.1) can be made very small (in our case it was of the order 10^{-14}). We used this to check our procedures for evaluation of the LM series, and found excellent agreement. Another application is to estimate the accuracy of the RG-extrapolated TCSA. Table 4.1 shows the difference of the expectation value for the $\Phi_{1,3}$ operator between the TBA prediction and RG-extrapolated TCSA evaluation in units of m_1 for several values of the dimensionless volume parameter, $l = m_1 L$. The results show the efficiency of the RG-extrapolated TCSA. We also showed the difference between the RG-extrapolated TCSA value and the one evaluated at the highest value of the cutoff; as exemplified by the data ate $l = 15$, the extrapolation results of an improvement of accuracy of almost two orders of magnitude for volumes $l \gtrsim 10$.

l	0.2	0.5	1	2	4	6	15
δf_{13}^{RG}	$8.6 \cdot 10^{-11}$	$1.2 \cdot 10^{-8}$	$5.3 \cdot 10^{-8}$	$1.7 \cdot 10^{-7}$	$5.5 \cdot 10^{-7}$	$1.5 \cdot 10^{-6}$	$3.6 \cdot 10^{-6}$
$\delta f_{13}^{e_{cut}=30}$	$8.6 \cdot 10^{-11}$	$4.1 \cdot 10^{-9}$	$1.0 \cdot 10^{-7}$	$9.0 \cdot 10^{-7}$	$6.5 \cdot 10^{-6}$	$1.9 \cdot 10^{-5}$	$1.9 \cdot 10^{-4}$

Table 4.1: The difference $\delta f_{13} = m_1^{6/7} \left(\langle \Phi_{1,3} \rangle_{\mathrm{TBA}} - \langle \Phi_{1,3} \rangle_{\mathrm{TCSA}} \right)$ between the RG-extrapolated TCSA evaluation and the TBA prediction for the VEV of $\Phi_{1,3}$.

Now we turn to the evaluation of the LM-series for the $\Phi_{1,2}$ operator. At infinite volume only the infinite volume VEV (B.4) contributes to the LM-series, while for small volumes almost all terms are relevant. Their contributions are proportional to $\sim e^{-\mu L}$, where μ is the sum of the masses of the particles entering a given term. For every contribution there's a characteristic volume where it becomes relevant. We calculated the first seven finite volume correction terms: 1, 2, 11, 12, 111, 22 and 112, where the number of 1 and 2 means the number of particle with mass m_1 and m_2 participating in the given contributions. Table 4.2 shows the value of the RG-extrapolated TCSA evaluation and the LM-series with the mentioned contributions for $\Phi_{1,2}$, while Table 4.3 shows the difference of the LM-series from the RG-extrapolated TCSA data while adding more and more contributions.

For $l > 1$, there is a steady improvement as more and more terms are added, and it is clear from the table that contributions from higher states are switched on at progressively lower volumes. Comparing the columns labeled +111 and +12, it can be seen that adding the 111 contribution makes the agreement of the LM series with TCSA worse; the reason is that the term 22 is of the same order of magnitude as the contribution from 111, so consistency requires adding them to the series together. Indeed, the deviation reported in column +22 is smaller than the one in column+12.

l	0.5	1.	1.5	2.	2.5	3.
TCSA	1.0776	1.5704	1.8957	2.0949	2.2060	2.2640
LM	1.3853	1.6215	1.9036	2.0960	2.2061	2.2640
l	3.5	4.	4.5	5.	7.	10.
TCSA	2.2936	2.3086	2.3163	2.3203	2.3246	2.3251
LM	2.2936	2.3086	2.3163	2.3203	2.3247	2.3251

Table 4.2: The values for $im_1^{4/7} \langle \Phi_{1,2} \rangle$ from the RG-extrapolated TCSA evaluation and the Leclair-Mussardo series.

For $l \lesssim 1$, the series does not converge very well; indeed for $l = 0.5$, there is no sign of any convergence. However, for such small values of the volume higher terms of the LM series would still be significant. Indeed, the leading corrections following the 112 term are the contributions1111 and 122, which can be estimated to be of order

$$e^{-4m_1 L} \quad \text{and} \quad e^{-(m_1+2m_2)L}$$

which at $l = 0.5$ give approximately 14% and 12%, respectively. This agrees well with the magnitude of the deviation at $l = 0.5$, as can also be seen from table 4.2.

For volumes $l < 3.5$ the estimated TCSA error is smaller than 10^{-6}, and so the deviation between TCSA and the LM series is dominated by the higher corrections to the LM series. However, the calculation of higher contributions becomes progressively slower as the number

l	0	+1	+2	+11	+12	+111	+22	+112
0.5	-1.2475	-0.7261	0.8304	0.7290	0.1691	0.1879	-0.4571	-0.3077
1	-0.7547	-0.4576	0.2609	0.2244	$6 \cdot 10^{-2}$	$7 \cdot 10^{-2}$	$-8 \cdot 10^{-2}$	$-5 \cdot 10^{-2}$
1.5	-0.4295	-0.2530	$8 \cdot 10^{-2}$	$7 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	$-1 \cdot 10^{-2}$	$-8 \cdot 10^{-3}$
2	-0.2302	-0.1259	$2 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	$4 \cdot 10^{-3}$	$5 \cdot 10^{-3}$	$-2 \cdot 10^{-3}$	$-1 \cdot 10^{-3}$
2.5	-0.1192	$-6 \cdot 10^{-2}$	$6 \cdot 10^{-3}$	$4 \cdot 10^{-3}$	$9 \cdot 10^{-4}$	$1 \cdot 10^{-3}$	$-3 \cdot 10^{-4}$	$-1 \cdot 10^{-4}$
3	$-6 \cdot 10^{-2}$	$-3 \cdot 10^{-2}$	$2 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$2 \cdot 10^{-4}$	$2 \cdot 10^{-4}$	$-4 \cdot 10^{-5}$	$-1 \cdot 10^{-5}$
3.5	$-3 \cdot 10^{-2}$	$-1 \cdot 10^{-2}$	$5 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	$3 \cdot 10^{-5}$	$4 \cdot 10^{-5}$	$-6 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$
4	$-2 \cdot 10^{-2}$	$-5 \cdot 10^{-3}$	$1 \cdot 10^{-4}$	$6 \cdot 10^{-5}$	$6 \cdot 10^{-6}$	$6 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$	$-1 \cdot 10^{-6}$
4.5	$-9 \cdot 10^{-3}$	$-2 \cdot 10^{-3}$	$4 \cdot 10^{-5}$	$1 \cdot 10^{-5}$	$2 \cdot 10^{-8}$	$1 \cdot 10^{-7}$	$-1 \cdot 10^{-6}$	$-1 \cdot 10^{-6}$
5	$-5 \cdot 10^{-3}$	$-9 \cdot 10^{-4}$	$1 \cdot 10^{-5}$	$3 \cdot 10^{-6}$	$-9 \cdot 10^{-7}$	$-9 \cdot 10^{-7}$	$-1 \cdot 10^{-6}$	$-1 \cdot 10^{-6}$
7	$-5 \cdot 10^{-4}$	$-3 \cdot 10^{-5}$	$-1 \cdot 10^{-6}$	$-1 \cdot 10^{-6}$	$-1 \cdot 10^{-6}$	$-1 \cdot 10^{-6}$	$-1 \cdot 10^{-6}$	$-1 \cdot 10^{-6}$
10	$-2 \cdot 10^{-5}$	$-2 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$
15	$8 \cdot 10^{-7}$	$9 \cdot 10^{-7}$	$9 \cdot 10^{-7}$	$9 \cdot 10^{-7}$	$9 \cdot 10^{-7}$	$9 \cdot 10^{-7}$	$9 \cdot 10^{-7}$	$9 \cdot 10^{-7}$

Table 4.3: The difference $\delta f_{12} = i m_1^{4/7} \left(\langle \Phi_{1,2} \rangle_{\rm LM} - \langle \Phi_{1,2} \rangle_{\rm TCSA} \right)$ between the RG-extrapolated TCSA evaluation and the Leclair-Mussardo series, depending on the multi-particle contributions included in the latter.

of particles to include increases (122 can be evaluated using a 10-particle form factor, cf. the remark at the end of Appendix B). Similarly to the case of 111 and 22, contributions 1111 and 122 are roughly of the same order and so they must be added to the series together. Treating the 10-particle form factor numerically proved to be rather difficult, so we have not evaluated the contribution 122.

However, there is a way to verify the consistency of the above considerations further. We fitted the deviation in the last column of Table 4.3 the difference from the TCSA by the Ansatz

$$a\, e^{-4m_1 L} + b\, e^{-(m_1+2m_2)L}$$

dictated by the particle content of the states 1111 and 122. As shown in figure 4.1, the fit was very successful. In addition, the parameters a and b turned out to be of the same order of magnitude, as expected from the above considerations.

For larger volumes the agreement cannot be improved by including other contributions, since it is dominated by the residual truncation error of the RG-extrapolated TCSA. We remark that a similar evaluation of the LM series for the case of $\Phi_{1,3}$ operator the agreement of the LM series and TBA calculations gave an agreement of precision 10^{-13} with the TBA for volumes $l > 7$. Figure 4.2 illustrates the improvement of the LM-series by adding more terms, as compared to the RG-extrapolated TCSA data.

5 A remark on finite volume form factors and μ-terms

In this section we digress to report an issue we found when we tested the accuracy of TCSA by comparing numerical results for the finite volume matrix elements to predictions of the finite volume form factor formalism, using the methods developed in [8, 9].

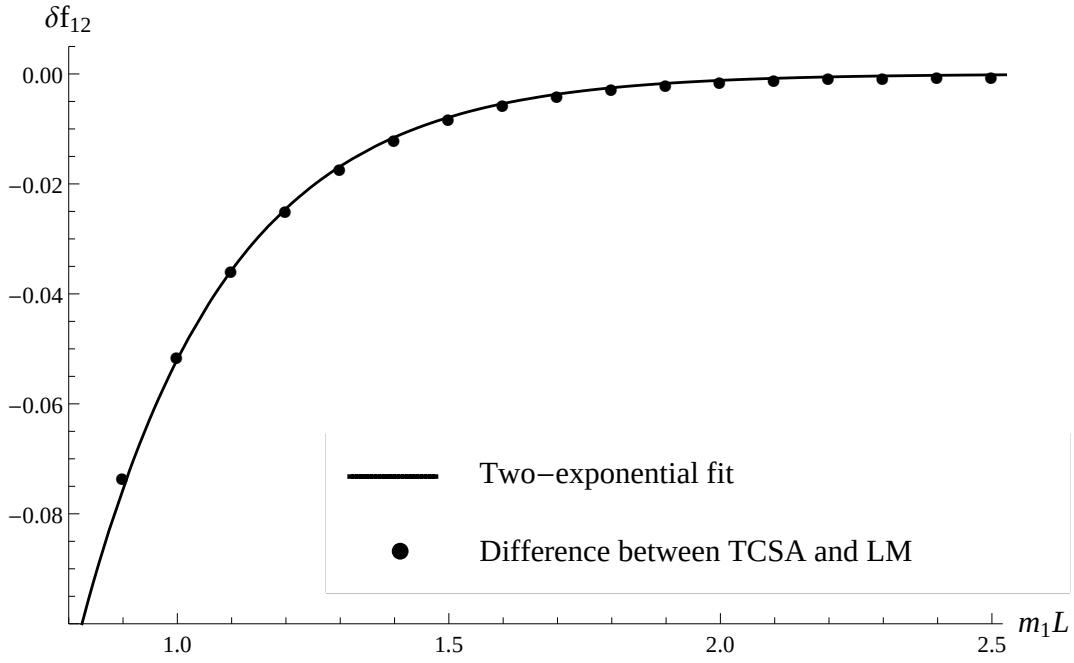

Figure 4.1: Fitting the residual difference δf_{12} between the LM series evaluated up to 112 and the RG extrapolated TCSA result

In [8] Pozsgay and Takács derived a formula for finite volume form factors valid up to exponential corrections:

$$
\begin{aligned}
{}_{j_1,\ldots,j_n}\langle\{J_1,\ldots,J_n\}|\,\mathcal{O}(0,0)\,|\{I_1,\ldots,I_m\}\rangle_{i_1,\ldots,i_m;L} = \\
\pm\frac{F^{\mathcal{O}}_{j_n,\ldots,j_1,i_1,\ldots,i_m}\left(\theta'_n+i\pi,\ldots,\theta'_1+i\pi,\theta_1,\ldots,\theta_m\right)}{\sqrt{\rho_{j_1,\ldots,j_n}\left(\theta'_1,\ldots,\theta'_n\right)\rho_{i_1,\ldots,i_m}(\theta_1,\ldots,\theta_m)}} \\
\times\Phi_{j_1,\ldots,j_n}\left(\theta'_1,\ldots,\theta'_n\right)^*\Phi_{i_1,\ldots,i_m}\left(\theta_1,\ldots,\theta_m\right) \qquad +O\left(e^{-\mu L}\right) \qquad (5.1)
\end{aligned}
$$

where I and J are the quantum numbers, i and j are the particle types in the given states, ρ is the density of sates and $F^{\mathcal{O}}$ is the infinite volume form factor, the phase factors Φ have the form

$$
\Phi_{i_1,\ldots,i_m}\left(\theta_1,\ldots,\theta_m\right) = \sqrt{\prod_{\substack{k,l=1\\k<l}}^{n} S_{i_k i_l}(\theta_k-\theta_l)}
$$

and the $\pm$ sign corresponds to the ambiguity in choosing the branch of the square root functions. We note that the same ambiguity is present in TCSA due to the fact that the choice of eigenvectors is not unique. Formula (5.1) is valid if there are no disconnected terms in the matrix element; for the treatment of disconnected terms cf. [9] .

The density of states in (5.1) can be obtained from the finite volume quantization conditions (valid up to exponential corrections)

$$
\begin{aligned}
Q^{(k)}_{i_1\ldots i_n}(\theta_1,\ldots,\theta_n) &= m_{i_k}L\sinh\theta_k+\sum_{j\neq k}\delta_{ij_i}\left(\theta_k-\theta_j\right)=2\pi I_k \qquad (5.2)\\
&\qquad I_k\in\mathbb{Z} \qquad , \qquad k=1,\ldots,n
\end{aligned}
$$

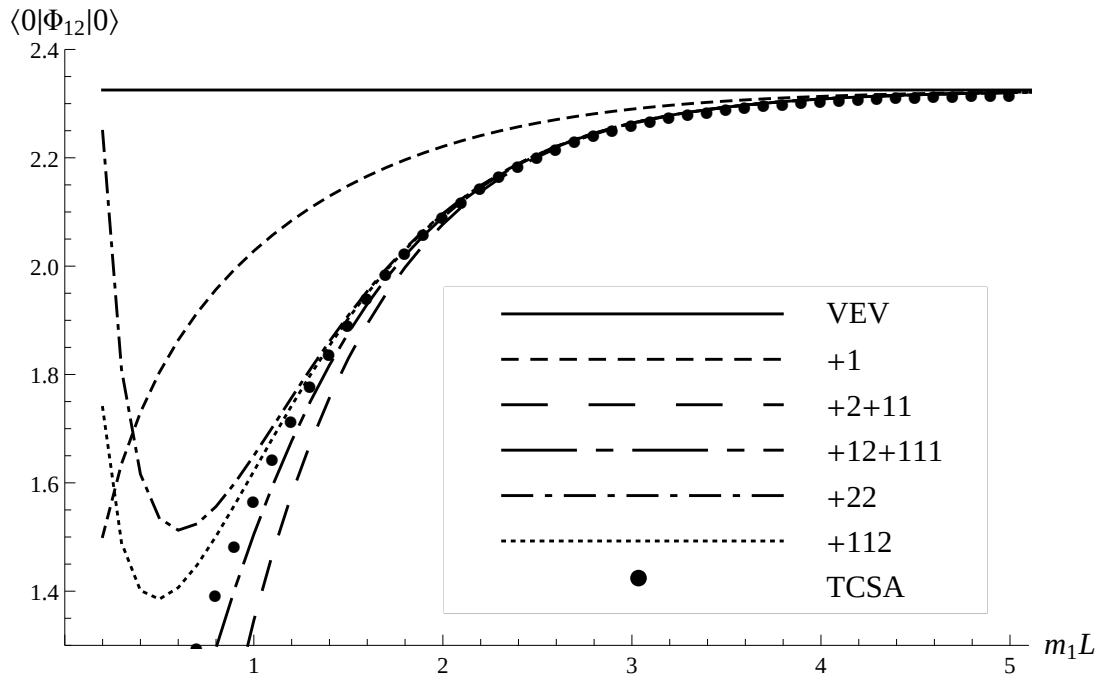

Figure 4.2: Leclair-Mussardo series against RG-extrapolated TCSA evaluation for the VEV of $\Phi_{1,2}$ by taking account more and more contributions, in units of m_1

as

$$\rho_{i_1\dots i_n}(\theta_1,\dots,\theta_n) = \det \mathcal{J}^{(n)} \qquad , \qquad \mathcal{J}^{(n)}_{kl} = \frac{\partial Q^{(k)}_{i_1\dots i_n}(\theta_1,\dots,\theta_n)}{\partial\theta_l} \quad , \quad k,l = 1,\dots,n \quad (5.3)$$

and it depends on the volume and it is positive for large volumes since its leading behavior is given by

$$\rho_{i_1,\dots,i_n}(\theta_1,\dots,\theta_n) = \prod_{k=1}^{n} m_{i_k} L\cosh\theta_k + O(L^{n-1})$$

where the subleading corrections depend on the scattering phase shifts [8]. Eventually, nothing prevents these factors from becoming zero or negative for smaller values of the volume, which would therefore produce a singularity in the finite volume form factor predicted by (5.1). Such singularities have not been encountered previously; however, in the T_2 model we found this behaviour for the state B_1B_2, as shown in figure 5.1.

On the other hand, the matrix elements calculated from TCSA prove to be regular (no singularities)[3]. Therefore we should take a closer look at the finite volume form factor formula (5.1). Since it is correct only up to exponential corrections, the natural idea is that exponential corrections could resolve the singularities.

In [22] Pozsgay introduced a method to describe a particular class of exponential corrections, the so-called μ-terms which originate from the composite structure of the particles under

[3]For a non-unitary model this is not necessarily the case, as the eigenvectors can have zero norm for some values of the volume. These norms appear in the denominator when evaluating the matrix elements in TCSA [8]. This does happen in the case of the boundary Lee-Yang model when the ground state level and the first excited level cross [26].

the bootstrap principle. Using the bound-state quantization first introduced in [27], these can be described by a continuation of the quantization relations (5.2) and the form factor formula (5.1) to complex rapidities. For more details we refer the reader to the original article [22], and to [28, 29] where this method was used with great success.

To demonstrate how the exponential corrections dissolve the singularities, let us examine the $|\{0,0\}\rangle_{12}$ state, which contains a B_1 and a B_2 particle, both with quantum number 0. The B_2 particle can be treated as the bound state of two B_1 particles which have complex rapidity. We can therefore treat the state as a three-B_1 state$|\{0,0,0\}\rangle_{111}$ and look for a solution of (5.2) with rapidities $\theta_1 = 0$, $\theta_2 = iu$ and $\theta_3 = -iu$. The single unknown u can be obtained by solving

$$im_1 L \sin u + \delta_{11}(iu) + \delta_{11}(2iu) = 0$$

and the analytic continuation of (5.1) (including the phase factors) gives the finite volume form factor including the μ-term contribution. Figure 5.1 shows the finite volume form factor with the naive evaluation and the correction by the μ-terms against the TCSA data for the vacuum-B_1B_2 matrix elements. It can be seen clearly that the exponential corrections do resolve the singularities, and the agreement between the TCSA data and form factor expression is very good down to small volumes where other exponential corrections become relevant.

Figure 5.2 shows the results for the B_1B_2- B_1B_2 diagonal matrix elements[4]. The diagonal matrix element of $\Phi_{1,3}$ operator contains no singularity, since it is related to the trace of the energy momentum tensor, and its diagonal matrix element can be expressed using the energy and momentum eigenvalues of the state as computed in the approximation given by (5.1), which are finite. However even in this case including the μ-terms improves the agreement with TCSA. For the operator $\Phi_{1,2}$ the conclusion is the same as in the non-diagonal case: the μ-terms again lead to a resolution of the singularity.

To end this section, we mention a singularity which is caused by the inclusion of μ-terms, in a marked contrast to the above two cases where the μ-terms eliminated singularities of the expression (5.1). Figure 5.3 shows the matrix elements multiplied by the one-particle density of states for the B_2 particle in the vacuum-B_2 case, i.e. the expression

$$\sqrt{\rho_2(0)}\langle 0|\mathcal{O}|\{0\}\rangle_2$$
$$\text{where} \quad \rho_2(0) = m_2 L$$

(the extra factor $\sqrt{\rho_2(0)}$ is put in to make the large volume asymptotics of the matrix elements a constant, which makes the plots easier to interpret). We see that including the μ-term correction, while improving the agreement with TCSA, also lead to a singularity at volume $l \sim 5.105$. It is plausible that other exponential corrections would resolve these singularities. The next class to be taken into account are the so-called F-terms, which would lead to a change in the quantization condition by terms that vanish exponentially with increasing volume. For the energy levels, such terms can be found from the excited TBA approach of [30, 31, 32]; a general treatment of F-terms is given in [33]. However, for the form factors the theoretical description of these corrections is not yet developed and is currently under active investigation [34].

[4]Note that these matrix elements contain disconnected contributions, and therefore must be computed by a formula different from (5.1); for details cf. [9, 22, 29].

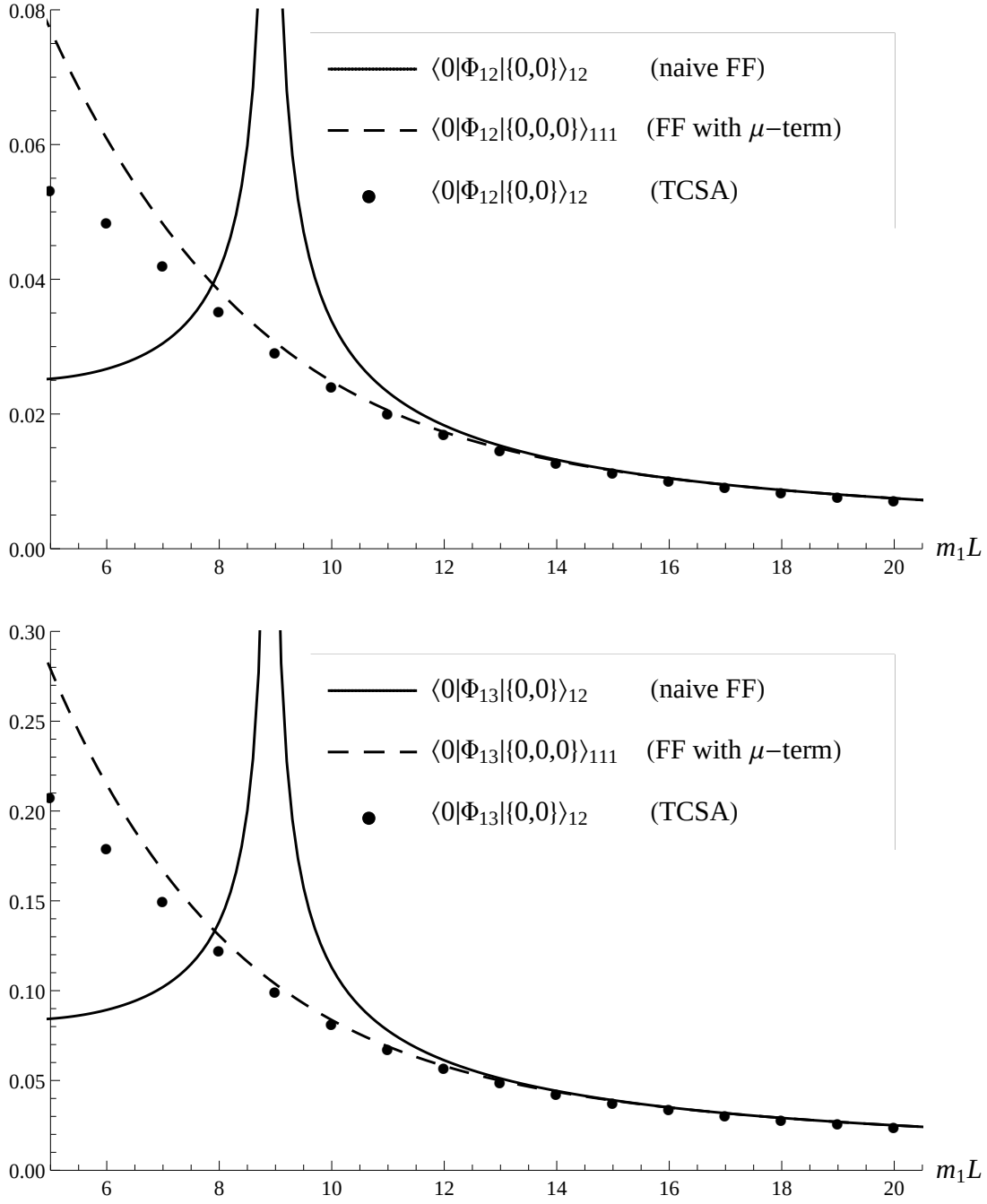

Figure 5.1: Vacuum-B_1B_2 matrix elements for $\Phi_{1,2}$ and $\Phi_{1,2}$ operators in units of m_1, where the solid line is the naive finite volume form factors result, the dashed line is the correction with the μ-terms and the dots represent the TCSA data. All data in the plot show absolute values of the matrix elements to get rid of the phase ambiguities related to choice of eigenvectors in TCSA.

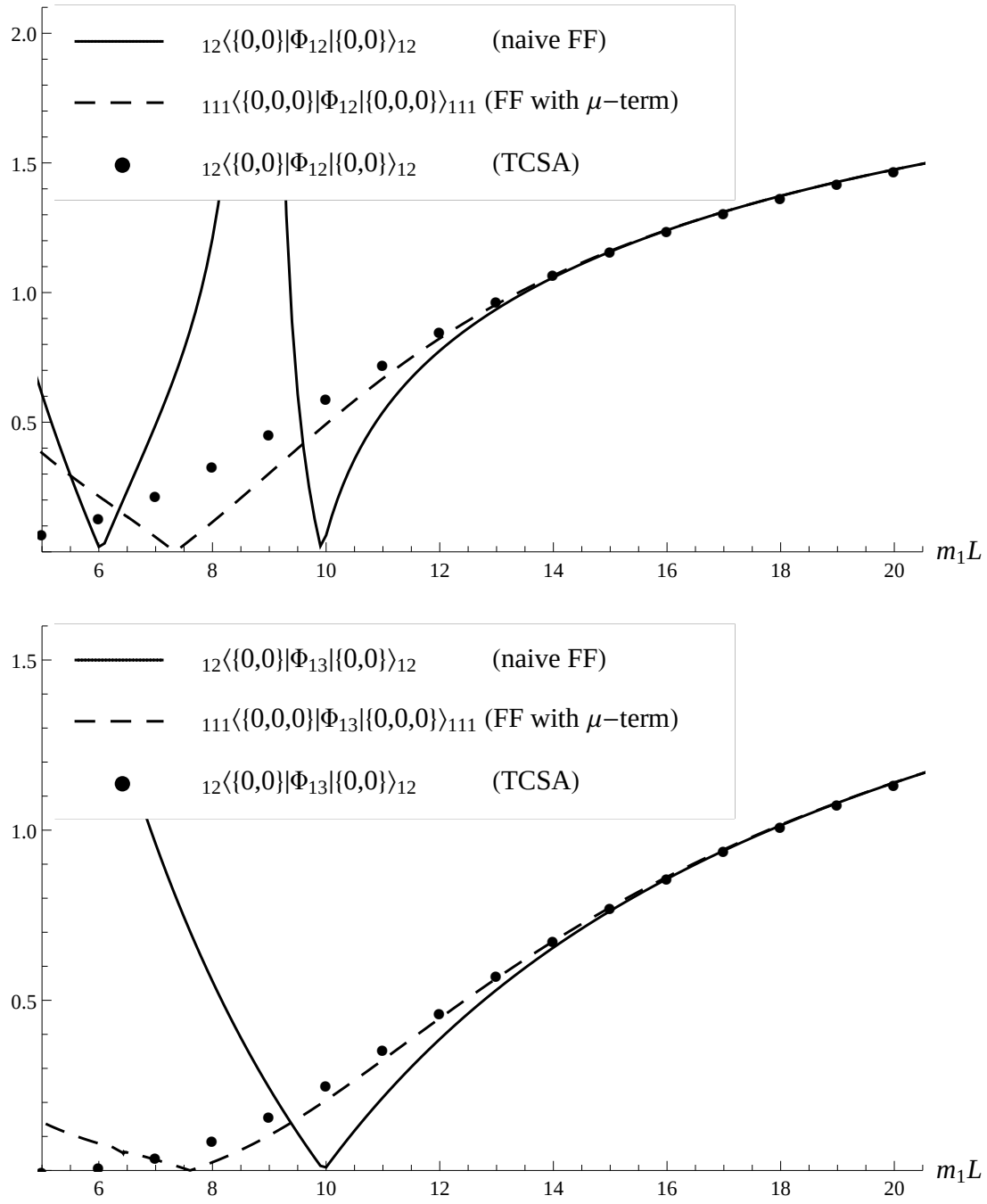

Figure 5.2: B_1B_2-B_1B_2 matrix elements for $\Phi_{1,2}$ and $\Phi_{1,2}$ operators in units of m_1, where the solid line is the naive finite volume form factors result, the dashed line is the correction with the μ-terms and the dots represent the TCSA data. The breaks in the lines are due to plotting the absolute values of the predicted matrix elements (the true value would become negative and the line cross below the real axis).

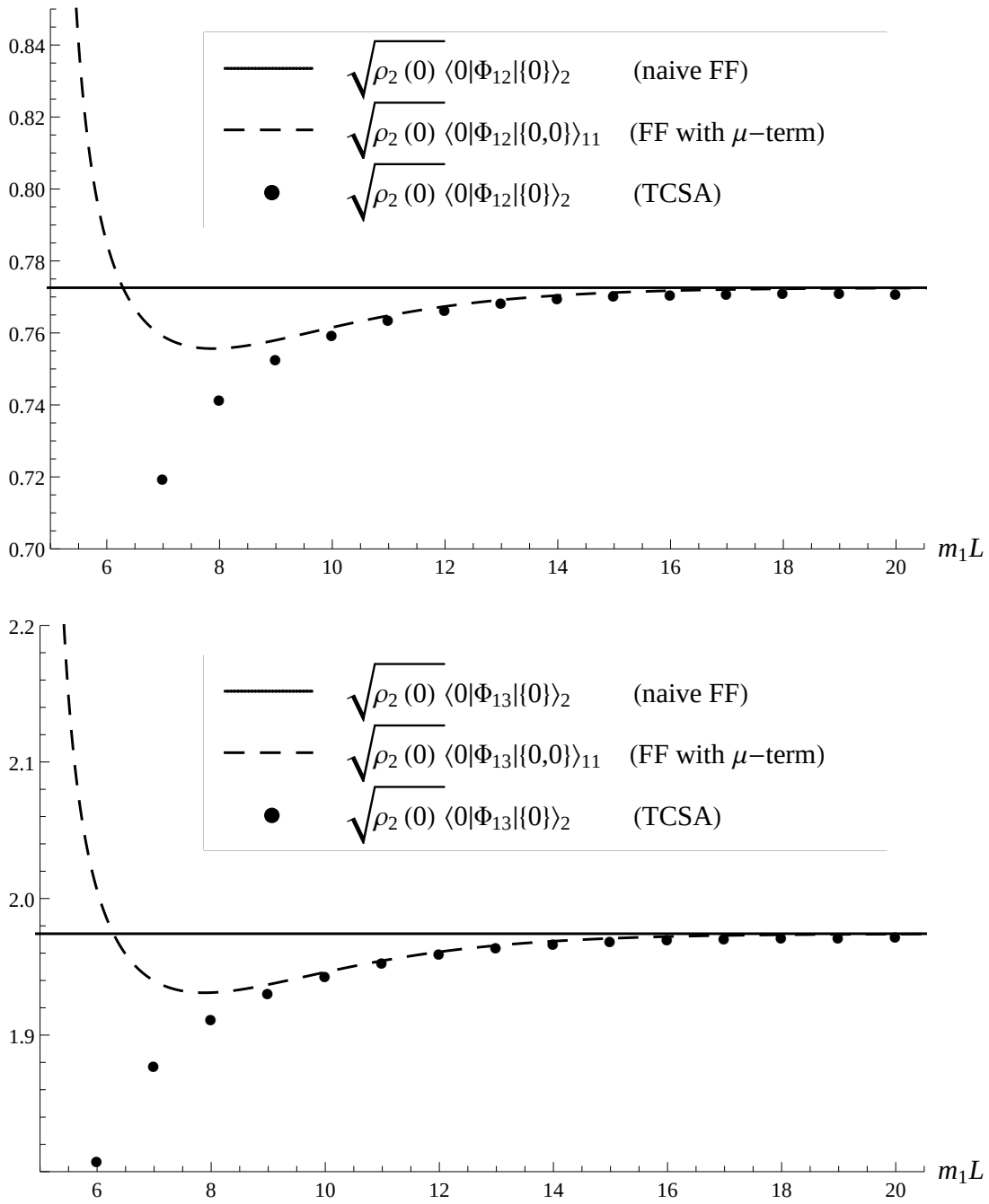

Figure 5.3: Vacuum-B_2 matrix elements multiplied by the one-particle density of states for the B_2 particle for $\Phi_{1,2}$ and $\Phi_{1,2}$ operators in units of m_1, where the solid line is the naive finite volume form factors result, the dashed line is the correction with the μ-terms and the dots represent the TCSA data.

6 Conclusions and outlook

In this paper we investigated expectation values of local operators in integrable quantum field theores, in finite volume or, equivalently, at finite temperature. We considered both numerical and analytical approaches, and established their consistency, thereby showing that a combination of these methods is able to give an accurate description of these expectation values for all scales from small to large volumes, i.e. from the high-temperature to the the low-temperature regime. For detailed studies we considered the so-called T_2 model, which has a nontrivial primary field besides the perturbing one, and all of its form factors are known explicitly to any number of particles.

The ultraviolet regime can be accessed using perturbed conformal field theory, and the non-perturbative truncated conformal space approach extends its validity significantly and into the infrared regime proper. When supplemented with renormalization group improvement, it can be matched to methods based on the infrared date (i.e. scattering theory) with an astonishing precision in a wide volume/temperature range (indeed, as shown under 3.3.3 and 3.3.4, the RG improvement is mandatory for a proper agreement). The infrared regime can be accessed using the Leclair-Mussardo series, which for the perturbing operator is equivalent to the TBA equation. We emphasize that the matching also works for operators for which the LM series cannot be derived in the TBA framework, as exemplified by the $\Phi_{1,2}$ operator in the T_2 model.

In addition to the above main results, the paper also contains a derivation of the cut-off dependence of expectation values in TCSA, which has already been used before [20], but without the formalism presented; the details are important in order to facilitate the use of this method in other calculations. Finally, while checking the accuracy of TCSA and the details of the T_2 form factors (B.2) using the methods developed in [8, 9], we found interesting cases in which the finite volume form factor formula, which neglects corrections decaying exponentially with the volume, predicts singularities that are not observed in TCSA and have no theoretical reason to exist. It was shown how these singularities are resolved by including the class of exponential corrections called μ-terms. However, somewhat surprisingly, this leads to the appearance of new singularities, which are in turn expected to be cured by higher exponential corrections. These higher exponential are not yet understood and it is clear that a systematic description of exponential corrections to matrix elements in finite volume is necessary to settle all these issues. Such a description can also have other applications in understanding the volume/temperature dependence of physical quantities. Work is in progress in this direction.

Acknowledgments

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A TCSA in the T_2 model

The minimal models $\mathcal{M}_{2,2n+3}$ have the central charge [35]

$$c = -\frac{2n(5+6n)}{3+2n}$$

When perturbed by the relevant operator $\Phi_{1,3}$ with conformal dimensions

$$h_{1,3} = \bar{h}_{1,3} = \frac{1-2n}{3+2n}$$

for an appropriate choice of the coupling constant they give rise to the massive quantum field theories T_n; the $n = 1$ case (T_1) is the scaling Lee-Yang model. For $n = 2$ the central charge is

$$c = -\frac{68}{7}$$

and there are two nontrivial primary field $\Phi_{1,2}$ and $\Phi_{1,3}$ with conformal weights $h_{1,2} = \bar{h}_{1,2} = -2/7$ and $h_{1,3} = \bar{h}_{1,3} = -3/7$. We consider the theory on a Euclidean space-time cylinder of circumference$L$ which can be mapped unto the punctured complex plane using

$$w = \exp\frac{2\pi}{L}(\tau - ix) \qquad , \qquad \bar{w} = \exp\frac{2\pi}{L}(\tau + ix) \tag{A.1}$$

under which primary fields transform as

$$\Phi(\tau, x) = \left(\frac{2\pi w}{L}\right)^{h} \left(\frac{2\pi \bar{w}}{L}\right)^{\bar{h}} \Phi(w, \bar{w})$$

The fields $\Phi_{1,k}$ are normalized with as

$$\langle 0|\Phi_{1,k}(w,\bar{w})\Phi_{1,k}(0,0)|0\rangle = \frac{1}{w^{h_{1,k}}\bar{w}^{\bar{h}_{1,k}}} \tag{A.2}$$

and the Hilbert space is given by

$$\mathcal{H}_{T_2} = \bigoplus_{h=0,-2/7,-3/7} \mathcal{V}_h \otimes \bar{\mathcal{V}}_h$$

where $\mathcal{V}_h$ ($\bar{\mathcal{V}}_h$) denotes the irreducible representation of the left (right) Virasoro algebra with highest weight h.

The Hamiltonian of the T_2 model takes the following form in the perturbed conformal field theory framework:

$$H = H_0 - \lambda \int_0^L dx \Phi_{1,3}(0, x) \tag{A.3}$$

where

$$H_0 = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$$

is the conformal Hamiltonian. When $\lambda > 0$ the theory above has two particles in its spectrum with masses m_1 and

$$m_2 = 2m_1 \cos\frac{\pi}{5} \tag{A.4}$$

The mass gap m_1 can be related to the coupling constant as [36]

$$\begin{aligned} \lambda &= \kappa m_1^{20/7} \\ \text{where} \qquad & \kappa = 0.04053795542\ldots \end{aligned} \tag{A.5}$$

and the bulk energy density is given by

$$\mathcal{B} = -\frac{m_1^2}{8\sin\frac{2\pi}{5}} \tag{A.6}$$

Due to translational invariance of the Hamiltonian (A.3), the conformal Hilbert space $\mathcal{H}$ can be split into sectors characterized by the eigenvalues of the total spatial momentum

$$P = \frac{2\pi}{L}\left(L_0 - \bar{L}_0\right)$$

where the operator $L_0 - \bar{L}_0$ generates spatial translations and its eigenvalue is called the conformal spin. For a numerical evaluation of the spectrum, the Hilbert space is truncated by imposing a cut in the conformal energy. The truncated conformal space corresponding to a given truncation and fixed value s of the Lorentz spin reads

$$\mathcal{H}_{\mathrm{TCS}}(s, e_{\mathrm{cut}}) = \left\{ |\psi\rangle \in \mathcal{H} \mid \left(L_0 - \bar{L}_0\right)|\psi\rangle = s|\psi\rangle,\ \left(L_0 + \bar{L}_0 - \frac{c}{12}\right)|\psi\rangle = e|\psi\rangle\ :\ e \le e_{\mathrm{cut}} \right\}$$

On this subspace, the dimensionless Hamiltonian matrix can be written as

$$h_{ij} = \frac{2\pi}{l}\left(L_0 + \bar{L}_0 - \frac{c}{12} - \frac{\kappa l^{2-2h_{1,3}}}{(2\pi)^{1-2h_{1.3}}} G^{(s)-1} B^{(s)}\right) \tag{A.7}$$

where energy is measured in units of the particle mass m_1, $l = m_1 L$ is the dimensionless volume parameter,

$$G_{ij}^{(s)} = \langle i|j\rangle \tag{A.8}$$

is the conformal inner product matrix and

$$B_{ij}^{(s)} = \langle i|\Phi_{1,3}(w,\bar{w})|j\rangle|_{w=\bar{w}=1} \tag{A.9}$$

is the matrix element of the operator Φ at the point $w = \bar{w} = 1$ on the complex plane between vectors $|i\rangle$, $|j\rangle$ from $\mathcal{H}_{\mathrm{TCS}}(s, e_{\mathrm{cut}})$. The natural basis provided by the action of Virasoro generators is not orthonormal and therefore $G^{(s)-1}$ must be inserted to transform the left vectors to the dual basis. The Hilbert space and the matrix elements are constructed using an algorithm developed by Kausch et al. [27].

Diagonalizing the matrix h_{ij} we obtain the energy levels as functions of the volume, with energy and length measured in units of m. We considered sectors with $s = 0, 1, 2, 3$, and the maximum value of the cutoff e_{cut} was 30, in which case the Hilbert space contains around twelve thousand vectors for even, and nine thousand vectors for odd values of the conformal spin. Once the eigenvectors are obtained, the matrix elements of local operators can be computed using the exponential mapping to evaluate matrix elements; for details cf. [8].

B Scattering theory and form factors of the T_2 model

The form factors of T_n models were constructed in [37] using the fact that they can be obtained as reductions of sine-Gordon theory at a particular value of the coupling [38]. In these particular models, only the breather states are retained in the spectrum. For T_2 only the first

two breathers B_1 and B_2 are retained in the spectrum, and their two-particle S matrices take the form

$$\begin{aligned} S_{11}(\theta) &= \left\{\frac{2}{5}\right\}_\theta \qquad S_{12}(\theta) = \left\{\frac{1}{5}\right\}_\theta \left\{\frac{3}{5}\right\}_\theta \qquad S_{22}(\theta) = \left\{\frac{2}{5}\right\}_\theta^2 \left\{\frac{4}{5}\right\}_\theta \\ \text{with} \qquad & \{x\}_\theta = \frac{\sinh\theta + i\sin\pi x}{\sinh\theta - i\sin\pi x} \end{aligned} \tag{B.1}$$

The form factors of primary fields with the fundamental particle can be obtained from the form factors of exponential operators in sine-Gordon theory with the first breather. The latter are given by

$$\begin{aligned} F^a_{\underbrace{11\ldots1}_{n}}(\theta_1,\ldots,\theta_n) &= \left\langle 0 \left| \mathrm{e}^{ia\beta\Phi(0)} \right| B_1(\theta_1)\ldots B_1(\theta_n) \right\rangle \\ &= \mathcal{G}_a(\xi)\,[a]_\xi\,(i\bar\lambda(\xi))^n \prod_{i<j} \frac{f_\xi(\theta_j-\theta_i)}{\mathrm{e}^{\theta_i}+\mathrm{e}^{\theta_j}}\, Q_a^{(n)}\left(\mathrm{e}^{\theta_1},\ldots,\mathrm{e}^{\theta_n}\right) \end{aligned} \tag{B.2}$$

where the parameter ξ is

$$\xi = \frac{\beta^2}{8\pi-\beta^2}$$

and

$$\begin{aligned} Q_a^{(n)}(x_1,\ldots,x_n) &= \det\left[a+i-j\right]_\xi \sigma^{(n)}_{2i-j}(x_1,\ldots,x_n)_{i,j=1,\ldots,n-1} \text{ if } n\geq 2 \\ Q_a^{(1)} &= 1 \quad , \qquad [a]_\xi = \frac{\sin\pi\xi a}{\sin\pi\xi} \\ \bar\lambda(\xi) &= 2\cos\frac{\pi\xi}{2}\sqrt{2\sin\frac{\pi\xi}{2}}\exp\left(-\int_0^{\pi\xi}\frac{dt}{2\pi}\frac{t}{\sin t}\right) \end{aligned}$$

with the function

$$\begin{aligned} f_\xi(\theta) &= v(i\pi+\theta,-1)v(i\pi+\theta,-\xi)v(i\pi+\theta,1+\xi) \\ &\quad v(-i\pi-\theta,-1)v(-i\pi-\theta,-\xi)v(-i\pi-\theta,1+\xi) \\ v(\theta,\zeta) &= \prod_{k=1}^{N}\left(\frac{\theta+i\pi(2k+\zeta)}{\theta+i\pi(2k-\zeta)}\right)^k \exp\left\{\int_0^\infty\frac{dt}{t}\Big(-\frac{\zeta}{4\sinh\frac{t}{2}}-\frac{i\zeta\theta}{2\pi\cosh\frac{t}{2}}\right. \\ &\quad \left.+\left(N+1-N\mathrm{e}^{-2t}\right)\mathrm{e}^{-2Nt+\frac{it\theta}{\pi}}\frac{\sinh\zeta t}{2\sinh^2 t}\Big)\right\} \end{aligned} \tag{B.3}$$

giving the minimal B_1B_1 form factor[5], while $\sigma_k^{(n)}$ denotes the elementary symmetric polynomial of n variables and order k defined by

$$\prod_{i=1}^n(x+x_i) = \sum_{k=0}^n x^{n-k}\sigma_k^{(n)}(x_1,\ldots,x_n)$$

[5]The formula for the function v is in fact independent of N; choosing N large extends the width of the strip where the integral converges and also speeds up convergence.

and $\mathcal{G}_a(\beta)$ is the vacuum expectation value of the field which is known exactly [39]. Form factors of higher breathers can be computed by representing them as bound states of B_1 particles; a useful formula for their evaluation can be found in Appendix A of [40].

The models T_n correspond to restriction at the coupling

$$\xi = \frac{2}{2n+1}$$

For the T_2 model, the restricted spectrum is composed of the first and the second breathers B_1 and B_2. The form factors of the operators $\Phi_{1,2}$ and $\Phi_{1,3}$ can be obtained as the cases $a = 1$ and $a = 2$ from formula (B.2) [37]; however, the vacuum expectation value needs to be replaced by the exact vacuum expectation value of the minimal model fields, derived in [41]:

$$\begin{aligned} \langle \Phi_{1,2} \rangle &= -2.3251365527 \cdots \times i\, m_1^{-4/7} \\ \langle \Phi_{1,3} \rangle &= 2.2695506880 \cdots \times m_1^{-6/7} \end{aligned} \tag{B.4}$$

A form factor containing n B_1 particles and m B_2 particles can be evaluated using a fundamental form factor (B.2) containing $n + 2m$ B_1 particles; therefore the connected form factor in (2.3) can be evaluated from a $2n + 4m$-particle fundamental form factor.

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